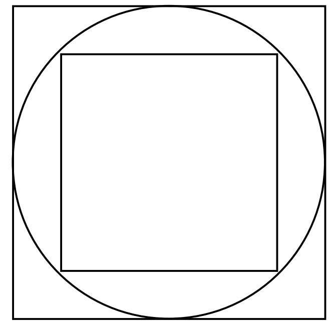
Circle and Squares

This problem gives you the chance to:
- calculate ratios of nested circles and squares

This diagram shows a circle with one square inside and one square outside.



The circle has radius *r* inches.

1. Write down the side of the large square in terms of *r*. __________ inches

2. Find the side of the small square in terms of *r*. __________ inches
   Show your work.

3. What is the ratio of the areas of the two squares? ______________________
   Show your work.

4. Draw a second circle inscribed inside the small square.
   Find the ratio of the areas of the two circles. ______________________
   Show your work.

8

Circles and Squares		Rubric
The core elements of performance required by this task are: • calculate ratios of nested circles and squares Based on these, credit for specific aspects of performance should be assigned as follows	points	section points
1. Gives correct answer: **2r**	1	1
2. Gives correct answer: $\sqrt{}$**2r** or 1.4r Uses the Pythagorean rule or trig ratios	1 1	2
3. Gives correct answer: **2** or 1/2 Shows work such as: $(2r)^2 \div (\sqrt{2}r)^2$	1 1	2
4. Gives correct answer: **2** or 1/2 Shows some correct work such as: the radius of the small circle is $r/\sqrt{2}$ the areas of the large and small circles are πr^2 and $\pi r^2/2$	1 2	3
Total Points		**8**

Circles and Squares

Work the task and look at the rubric? What are the mathematical demands needed to interpret the diagrams? To solve the ratios? ___________________________________

____

____

____

Look at student work on interpreting the diagrams part 1 and 2. For part 2 how many of your students put:

$\sqrt{2}$**r** or 1.4r	r	r/$\sqrt{2}$**r**	r^2	No work	And expression without r	Other

Look at student work for part 3, ratio of the squares. How many of your students put:

2	2/3	1/4	expression with r's	No work	Other

What were some of the problems that you saw in student work? What properties were they not paying attention to?

Finally look at work on the ratio of circles. How many of your students put:

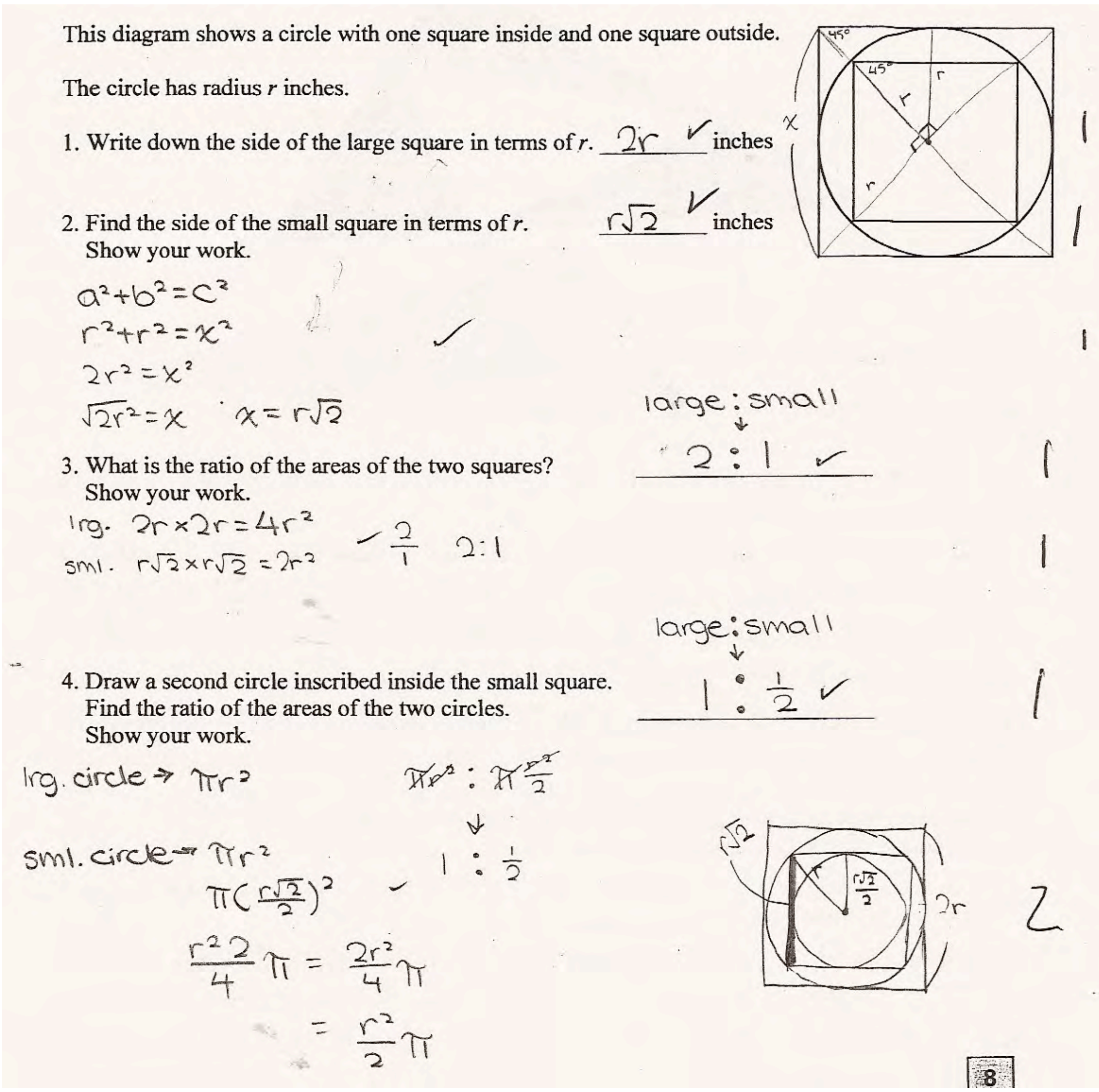
- Correct answer?___________ Now look at the reasoning. Did the reasoning follow the problem?______________ or did they just luck into an answer?
- How many of the students thought the ratio was 1/4?_______
- How many ratios still had variables or π?________________

What mathematical ideas were students struggling with as you examined their thinking?

Looking at Student Work on Circles and Squares

Student A makes good use of the diagram, adding extra lines, angle measurements, and dimensions to help think about the task. Notice that in part 4 the student creates an additional diagram to help think out the implications of the new circle. In part 2 the use of Pythagorean theorem is clearly laid out to find the side size of the small square. The student uses labels to find the areas to make it clearer how the ratios are calculated.

Student A


This diagram shows a circle with one square inside and one square outside.

The circle has radius r inches.

1. Write down the side of the large square in terms of r. $2r$ ✓ inches

2. Find the side of the small square in terms of r. $r\sqrt{2}$ ✓ inches
   Show your work.

   $a^2+b^2=c^2$
   $r^2+r^2=x^2$ ✓
   $2r^2=x^2$
   $\sqrt{2r^2}=x$ $\quad x=r\sqrt{2}$

3. What is the ratio of the areas of the two squares? large : small $2:1$ ✓
   Show your work.

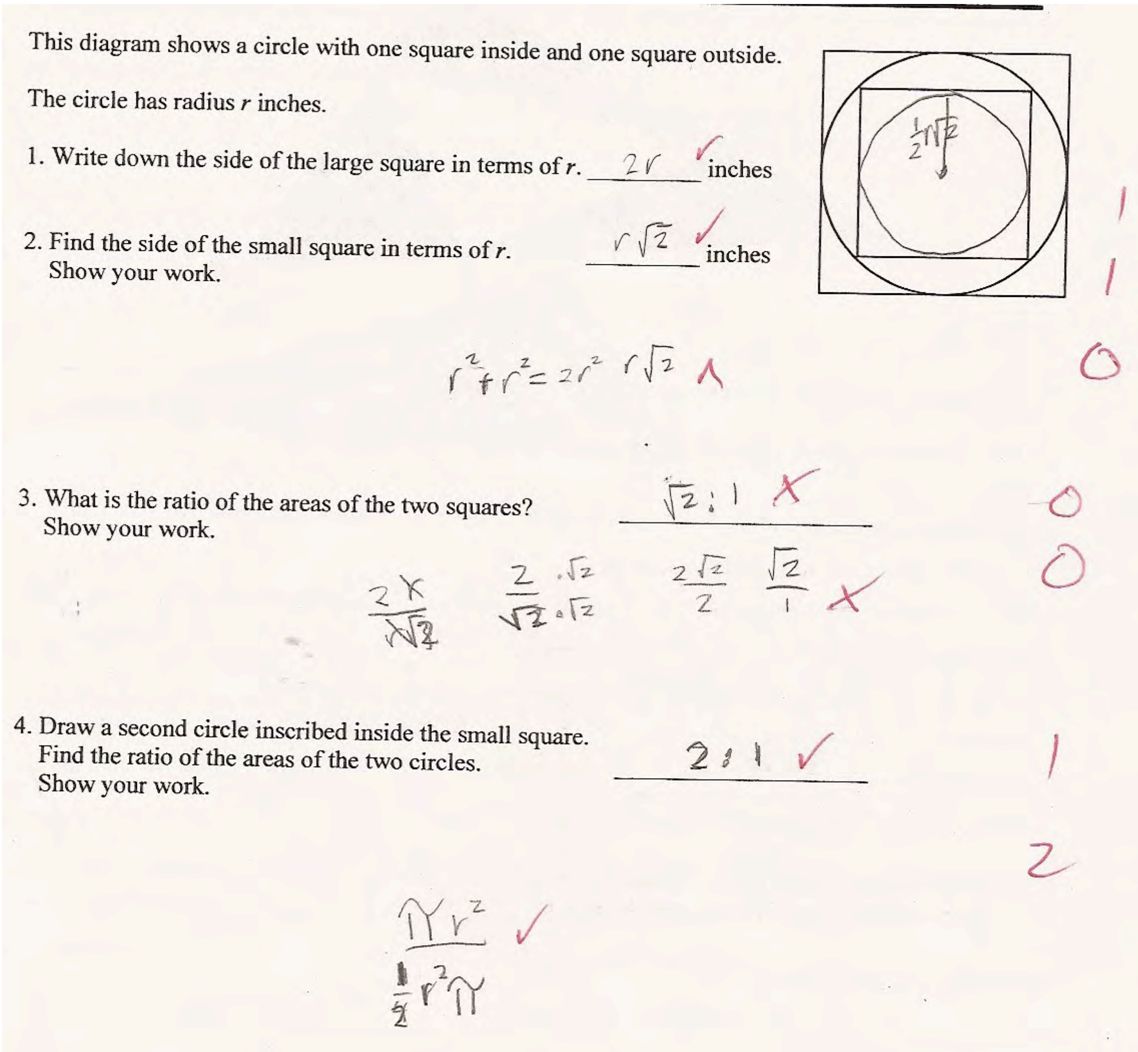
   lrg. $2r \times 2r = 4r^2$ ✓ $\frac{2}{1}$ $2:1$
   sml. $r\sqrt{2} \times r\sqrt{2} = 2r^2$

4. Draw a second circle inscribed inside the small square. large : small $1 : \frac{1}{2}$ ✓
   Find the ratio of the areas of the two circles.
   Show your work.

   lrg. circle → πr^2 $\quad \pi r^2 : \pi \frac{r^2}{2}$ → $1 : \frac{1}{2}$
   sml. circle → πr^2
   $\pi\left(\frac{r\sqrt{2}}{2}\right)^2$ ✓
   $\frac{r^2 2}{4}\pi = \frac{2r^2}{4}\pi$
   $= \frac{r^2}{2}\pi$

Student B is able to interpret the diagram and find the dimensions of the square. The student doesn't find the area in part 3 before forming the ratios. The student just gives a ratio of side lengths. *How and why does the ratio for the side length differ from the ratio for area? Have students done investigations to help them think about this big mathematical idea?*

Student B


This diagram shows a circle with one square inside and one square outside.

The circle has radius r inches.

1. Write down the side of the large square in terms of r. $2r$ ✓ inches

2. Find the side of the small square in terms of r. $r\sqrt{2}$ ✓ inches
Show your work.

$r^2 + r^2 = 2r^2 \quad r\sqrt{2}$

3. What is the ratio of the areas of the two squares? $\sqrt{2} : 1$ ✗
Show your work.

$\frac{2r}{r\sqrt{2}} \quad \frac{2}{\sqrt{2}} \cdot \frac{\sqrt{2}}{\sqrt{2}} \quad \frac{2\sqrt{2}}{2} \quad \frac{\sqrt{2}}{1}$ ✗

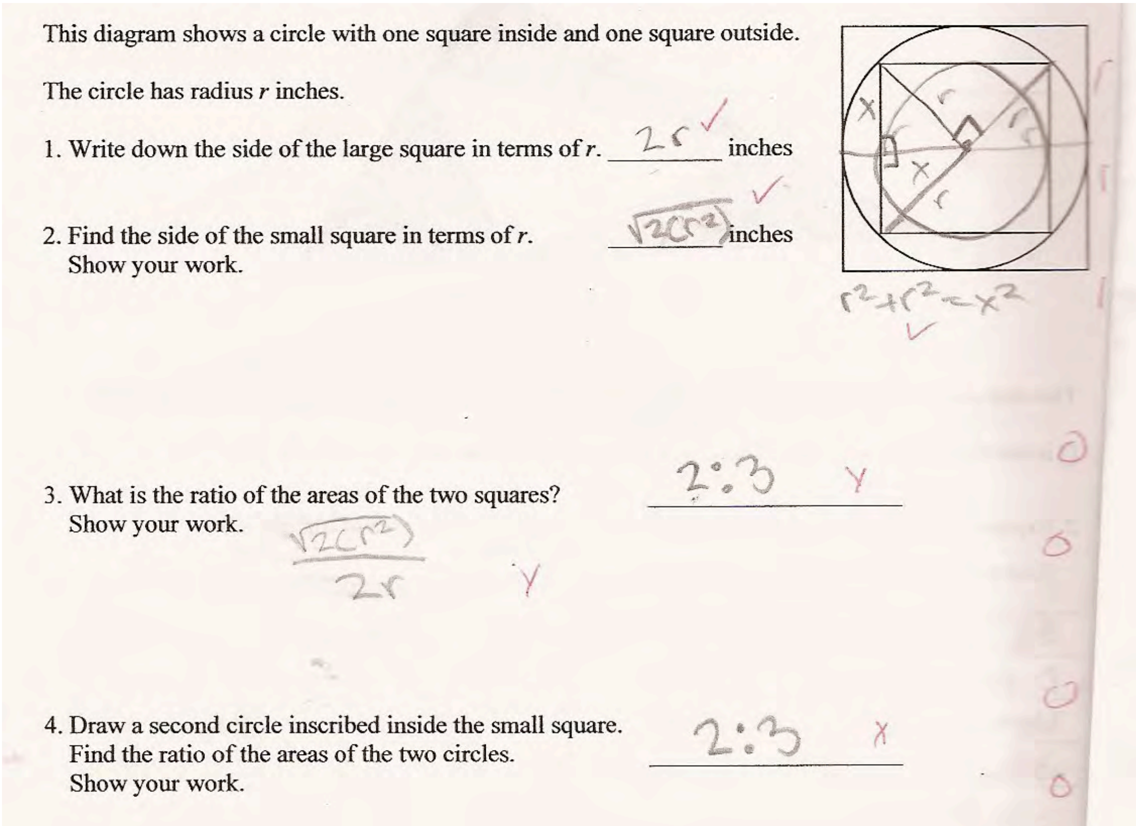
4. Draw a second circle inscribed inside the small square.
Find the ratio of the areas of the two circles. $2:1$ ✓
Show your work.

$\frac{\pi r^2}{\frac{1}{2} r^2 \pi}$ ✓

1
1
0
0
0
1
2

Student C is able to use the diagram to find the side lengths of the squares. The student's thinking breaks down in trying to work with the ratios. *Besides the idea that the student is looking at side length rather than area, how would you categorize or describe the other error the student makes?*

Student C



This diagram shows a circle with one square inside and one square outside.

The circle has radius *r* inches.

1. Write down the side of the large square in terms of *r*. $2r$ ✓ inches

2. Find the side of the small square in terms of *r*. $\sqrt{2(r^2)}$ ✓ inches
   Show your work.

$r^2+r^2=x^2$ ✓

3. What is the ratio of the areas of the two squares? 2:3
   Show your work.

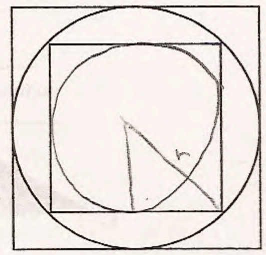
$\frac{\sqrt{2(r^2)}}{2r}$

4. Draw a second circle inscribed inside the small square.
   Find the ratio of the areas of the two circles. 2:3 ✗
   Show your work.

Student D makes in error in finding the side length of the small square. Had there been follow through, the student's work on ratio in part 3 would have been acceptable. *Can you speculate about the thinking in part 2?*

Student D

This diagram shows a circle with one square inside and one square outside.



The circle has radius *r* inches.

1. Write down the side of the large square in terms of *r*. $2r$ inches ✓

   $2r$

2. Find the side of the small square in terms of *r*. $\frac{r\sqrt{2}}{2}$ inches ✗
   Show your work.

   $\frac{r}{\sqrt{2}}$ $\frac{r\sqrt{2}}{2}$

   Big : Small

3. What is the ratio of the areas of the two squares? $4 : \frac{1}{2}$ or $8 : 1$ ✗
   Show your work.

   $4r^2 : \frac{r^2}{2}$ $4 : \frac{1}{2}$

   $\frac{r\sqrt{2}}{2} \cdot \frac{r\sqrt{2}}{2} = \frac{r^2 2}{4}$ $\frac{r^2}{2}$

4. Draw a second circle inscribed inside the small square. $1 : \frac{1}{2}$ or $2 : 1$ ✓
   Find the ratio of the areas of the two circles.
   Show your work.

   $\pi r^2 = A$

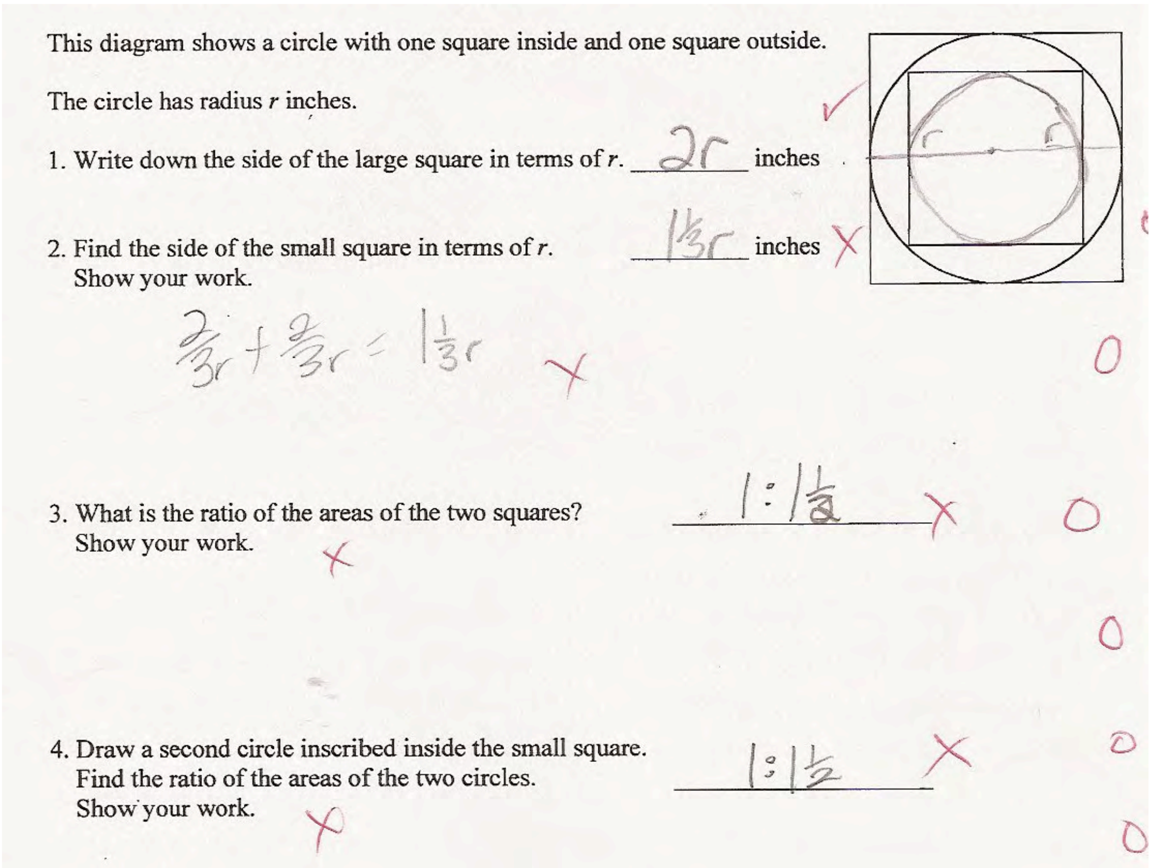
   $\pi\left(\frac{r\sqrt{2}}{2}\right)^2 = \pi \frac{r^2}{2}$ ✓

   $\pi r^2 : \frac{\pi r^2}{2}$ ✓

   $1 : \frac{1}{2}$

Student E is able to find the dimensions of the large square, but uses estimation to find the size of the small square. *What kinds of activities or discussions do we have with students to help them understand when to use an estimate and when an exact answer is expected? Can you figure out where the 1/12 comes from in the ratio?* While little work is shown, it seems that the student is again comparing sides rather than areas.

Student E



This diagram shows a circle with one square inside and one square outside.

The circle has radius *r* inches.

1. Write down the side of the large square in terms of *r*. 2r inches

2. Find the side of the small square in terms of *r*. $1\frac{1}{3}r$ inches
   Show your work.

   $\frac{2}{3}r + \frac{2}{3}r = 1\frac{1}{3}r$

3. What is the ratio of the areas of the two squares? $1 : 1\frac{1}{12}$
   Show your work.

4. Draw a second circle inscribed inside the small square.
   Find the ratio of the areas of the two circles. $1 : 1\frac{1}{2}$
   Show your work.

Geometry Task 5 Circle and Squares

Student Task	Calculate ratios of nested circles and squares.
Core Idea 4 Geometry and Measurement	**Analyze characteristics and properties of two- and three-dimensional shapes; develop mathematical arguments about geometric relationships; apply appropriate techniques, tools, and formulas to determine measurements.**
Core Idea 2 Mathematical Reasoning and Proofs	**Employ forms of mathematical reasoning and proof appropriate to the solution of the problem at hand, including deductive and inductive reasoning, making and testing conjectures and using counter examples and indirect proof.** • Show mathematical reasoning in solutions in a variety of ways including words, numbers, symbols, pictures, charts, graphs, tables, diagrams and models. • Identify, formulate and confirm conjectures. • Establish the validity of geometric conjectures using deduction; prove theorems, and critique arguments made by others.

The mathematics of this task:
- Reading and interpreting a diagram
- Using Pythagorean theorem to find the lengths of the legs given a hypotenuse
- Working with squares and square roots
- Finding area of squares and circles
- Making ratios

Based on teacher observations, this is what geometry students knew and were able to do:
- Find the side length of the large square
- Use Pythagorean theorem to find the side length of the small square

Areas of difficulty for geometry students:
- Using area to form ratios rather than side lengths
- Finding the size of the smaller radius

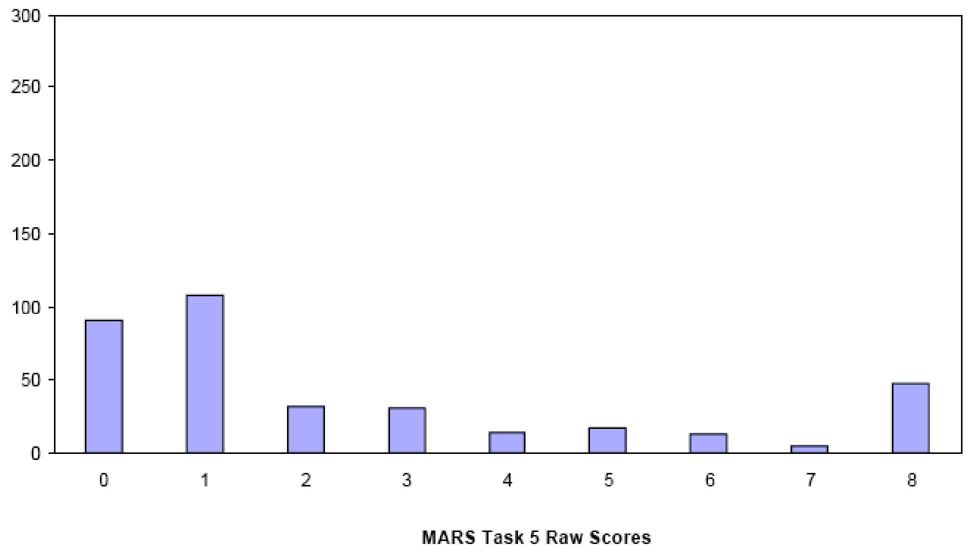
Task 5 - Circle & Squares

Mean: 2.49 StdDev: 2.72

Table 54: Frequency Distribution of MARS Test Task 5, Course 2

Task 5 Scores	Student Count	% at or below	% at or above
0	91	25.5%	100.0%
1	108	55.7%	74.5%
2	32	64.7%	44.3%
3	31	73.4%	35.3%
4	14	77.3%	26.6%
5	17	82.1%	22.7%
6	12	85.4%	17.9%
7	5	86.8%	14.6%
8	47	100.0%	13.2%

Figure 63: Bar Graph of MARS Test Task 5 Raw Scores, Course 2



The maximum score available on this task is 8 points.
The minimum score for a level 3 response, meeting standards, is 4 points.

Many students, 75%, could find the side length for the large square. Less than half the students, 35%, could also find the side length for the small square. Some students could find the length of the side for the large square, find the ratio of areas with missing or incorrect work, and find the ratio of the areas of the circles. 13% of the students could meet all the demands of the task including finding the length of the small square and using areas to find the ratio of the areas of the squares. More than 25% of the students scored no points on this task. All of the students in the sample with this score attempted the task.

Circle and Squares

Sample size for geometry was very small.

Points	Understandings	Misunderstandings
0	All the students in the sample with this score attempted the task.	Students had difficulty finding the dimensions for the large square. Common errors were $(2r)^2$ or $4r^2$.
1	Students could interpret the diagram to find the side length of the large square.	Students could not find the dimension of the small square. About 8% of the students thought it was r. About 4% thought it was $4/\sqrt{2}$. About 3% thought it was r^2.
3	Students could find the side lengths for both squares.	Students had difficulty with the ratios of the area of the circles. About 7% showed no work for this part. About 7% thought the ratio was 1/4. About 16% had expressions with π or r.
4	Students could find the dimension of the large square, give an answer for the ratio in part 3 with missing or incorrect work, and find the ratio and show correct work for the area of the circles.	Students often compared side lengths instead of areas in part 3. 5% of the students thought the ratio was 1:4. 3% thought the ratio was 2/3. 7% of the students did not attempt part 3.
8	Students could read and interpret a diagram to find the side length of a large square in terms of the radius of an inscribed circle. They could use Pythagorean theorem to find the length of the small square given the hypotenuse. Students could compare the ratios of the areas of circles and squares.	

Implications for Instruction

Students at this grade level need practice using diagrams as tools for solving problems. They should mark down what they know and what they need to find out. Students did not look at the correct parts of the diagram or think about the relationships needed to quantify some of the side lengths. Students need to be able to compose and decompose geometric shapes. Students should be comfortable adding lines where necessary to help find needed information.

Students still do not recognize situations where Pythagorean theorem should be applied.

Reflecting on the Results for Geometry as a Whole:

Think about student work through the collection of tasks and the implications for instruction. What are some of the big misconceptions or difficulties that really hit home for you?

If you were to describe one or two big ides to take away and use for planning for next year, what would they be?

What are some of the qualities that you saw in good work or strategies used by good students that you would like to help other students develop?

Four areas that stand out for the Collaborative as a whole are:

1. Using algebra to express ideas: Students had trouble expressing ideas in Numbers. They were clear how to use algebra to describe an even number. They didn't know how to express two numbers using the same variable by quantifying their difference. Students did not concentrate on all the details of the pattern before trying to make a generalization, often because they were too focused on the outcome.
2. Composing and decomposing shapes: In Glasses students had difficulty recognizing a half sphere in context. Students also confused the height of the hemisphere with part of the length of the cylinder. Some students confused half the height with half the volume for irregularly shaped objects. Other students did not realize that objects fill from the bottom up.

3. Making a complete mathematical justification: In both Pentagon and Triangles students had trouble developing the complete logic of a justification. They would often take the proof statement as one of the givens. They might make assumptions based on how something looked in a diagram, rather than proving the relationship from the known facts. They would present some reasonable ideas, but not finish with how it relates back to the statement that they are justifying.
4. Reading and Interpreting Diagrams: Students had difficulty interpreting the diagrams in Triangles, often picking an incorrect height because of the orientation of the triangle on the page. Students could not match angles when checking for similarity even though the triangles were embedded within each other. In Circle and Squares students did not see how the radius related to finding the side length of the small square. Where successful students tended to add lines to help make sense of the diagram or clarify ideas, less successful students did not mark on their diagrams.

Performance Assessment Task

Circle and Squares
Grade 10

This task challenges a student to analyze characteristics of 2-dimensional shapes to develop mathematical arguments about geometric relationships. A student must find the sides of squares given the radius of an inscribed and circumscribed circle. The student must be able to compare the ratio of the area of the two squares and of the two circles.

Common Core State Standards Math - Content Standards

High School – Geometry – Circles
Understand and apply theorems about circles.
G-C.2 Identify and describe relationships among angles, radii, and chords. *Include the relationship between central, inscribed and circumscribes angles; inscribed angles on a diameter are right angles; the radius of a circle is perpendicular to the tangent where the radius intersects the circle.*

G-C.3 Construct the inscribed and circumscribed circles of a triangle, and prove properties of angles for a quadrilateral inscribed in a circle.

Common Core State Standards Math – Standards of Mathematical Practice

MP.1 Make sense of problems and persevere in solving them.
Mathematically proficient students start by explaining to themselves the meaning of a problem and looking for entry points to its solution. They analyze givens, constraints, relationships, and goals. They make conjectures about the form and meaning of the solution and plan a solution pathway rather than simply jumping into a solution attempt. They consider analogous problems, and try special cases and simpler forms of the original problem in order to gain insight into its solution. They monitor and evaluate their progress and change course if necessary. Older students might, depending on the context of the problem, transform algebraic expressions or change the viewing window on their graphing calculator to get the information they need. Mathematically proficient students can explain correspondences between equations, verbal descriptions, tables, and graphs or draw diagrams of important features and relationships, graph data, and search for regularity or trends. Younger students might rely on using concrete objects or pictures to help conceptualize and solve a problem. Mathematically proficient students check their answers to problems using a different method, and they continually ask themselves, "Does this make sense?" They can understand the approaches of others to solving complex problems and identify correspondences between different approaches.

MP.7 Look for and make use of structure.
Mathematically proficient students try to look closely to discern a pattern or structure. Young students, for example, might notice that three and seven more is the same amount as seven and three more, or they may sort a collection of shapes according to how many sides the shapes have. Later, students will see 7 x 8 equals the well-remembered 7 x 5 + 7 x 3, in preparation for learning about the distributive property. In the expression $x^2 + 9x + 14$, older students can see the 14 as 2 x 7 and the 9 as 2 + 7. They recognize the significance of an existing line in a geometric figure and can use the strategy of drawing an auxiliary line for solving problems. They also can step back for an overview and shift perspective. They can see complicated things, such as some algebraic expressions, as single objects or being composed of several objects. For example, they can see $5 – 3(x - y)^2$ as 5 minus a positive number times a square and use that to realize that its value cannot be more than 5 for any real numbers x and y.

Assessment Results

This task was developed by the Mathematics Assessment Resource Service and administered as part of a national, normed math assessment. For comparison purposes, teachers may be interested in the results of the national assessment, including the total points possible for the task, the number of core

points, and the percent of students that scored at standard on the task. Related materials, including the scoring rubric, student work, and discussions of student understandings and misconceptions on the task, are included in the task packet.				
Grade Level	Year	Total Points	Core Points	% At Standard
10	2008	8	4	27%